On the non-injective points in the image of a space-filling curve, II.

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Abstract

We show once again that the non-injective points in the image of a space-filling curve are dense and in fact that in each open neighborhood there are uncountably many non-injective points. All this is done using a technique suggested by A.G. Henriques.

1 Space-filling curves

Our aim is to simplify the proof of the theorem first written down in [1]. For the sake of completeness, we repeat the definitions used there. We start with the definition of the objects we study here:

Definition 1.1. Space-filling curve *We call a surjective continuous function $f : [0,1] \to [0,1]^n$ an **(n-dimensional) space-filling curve**.*

The points of self-intersection in the image of a curve will be called non-injective points:

Definition 1.2. Non-injective points *Let $f : U \to V$ be a function. We call $v \in V$ a **non-injective point (with respect to f)** if there exist $u, w \in U$ such that $u \neq w$ and $f(u) = f(w) = v$. That is, if at least two distinct points in the domain of f are both mapped to v.*

In the case of a curve, these are the points of self-intersection.

2 Formulation and proof of the theorem

Using the definitions of the last section, we formulate a stronger version of the theorem in [1].

Theorem 2.1. *There are uncountably many non-injective points of a n-dimensional space-filling curve in each open neighborhood of $[0,1]^n$.*

We will prove this theorem using topological arguments, by looking at the boundary of the image of a certain subset of $[0,1]$ under a space-filling curve. This method was suggested by A.G. Henriques.

Proof. Let $f : [0,1] \to [0,1]^n$ be a space-filling curve and let $U \subset [0,1]^n$ be open. Then there is a open $V \subset U$ such that $V \cap \partial[0,1]^n = \emptyset$.

Let $W = f^{-1}(V)$, which is an open subset of $[0,1]$ and hence contains a closed interval K which can be chosen such that $f(K)$ is not a single point. Let $M = f(K)$, $N = f([0,1]\backslash K)$ and $N' = f(\overline{[0,1]\backslash K})$. Then all points in $M \cap N \subset V$ are non-injective points contained in V and $M \cap N'$ contains at most two points more that $M \cap N$. Thus, if we can show that $M \cap N'$ is uncountable, the theorem is proven.

Note that since f is a space-filling curve, $[0,1]^n \backslash M$ is contained in N and hence in N'. From this it follows that $\overline{[0,1]^n \backslash M} \subset N'$ since N' is closed as the image of a compact set. But $\partial M = \overline{[0,1]^n \backslash M} \cap \overline{M} = \overline{[0,1]^n \backslash M} \cap M$ since M is closed. Thus $\partial M \subset M \cap N'$.

It is now a simple task to establish that ∂M is uncountable. Consider $\pi_i(M)$ for π_i projection on the ith component. This is a connected subset of $[0,1]$ and since not all of them can be a single point (for then $f(K)$ would be a single point), we can without loss of generality assume that $\pi_1(M)$ is a nondegenerate interval L. Now consider for each $l \in L$ then $(n-1)$-dimensional hyperplane $\pi_1^{-1}(l)$. Since $V \cap \partial[0,1]^n = \emptyset$, it follows that $\pi_1^{-1}(l) \cap \partial[0,1]^n = \emptyset$ as well. But $\pi_1^{-1}(l) \cap M \neq \emptyset$ and hence $\pi_1^{-1}(l)$ contains at least one point of ∂M. Since L is uncountable, it follows that ∂M is uncountable. □

3 Consequences and questions

We have the following consequence, establishing an answer to a question asked by T.O. Rot.

Corollary 3.1. *The number of non-injective points of a space-filling curve is uncountably infinite.*

An extension to the other spaces which Hahn-Mazurkiewisz tells us are the image of a continuous curve is not obvious, since our proof uses some properties of $[0,1]^n$. The question is whether the alternative proved in [2], can be improved to say when the non-injective points are dense, they are uncountable in each open subset.

4 Conclusion

This proof greatly simplifies our earlier proof, however it is more difficult to generalize. It is stronger, in the sense that now we don't only know that the non-injective points in the image of a space-filling curve are dense, but also uncountable in each open subset.

References

[1] **Kupers, A.P.M.**, *On the non-injective points in the image of a space-filling curve*, 2008, HTTP://WWW.PHYS.UU.NL/∼3021009/.

[2] **Kupers, A.P.M.**, *On the non-injective points in the image of a continuous curve.*, 2008, HTTP://WWW.PHYS.UU.NL/∼3021009/.